Classical Exchange Algebra of the Superstring on S^5 with the AdS-Time

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Abstract

A classical exchange algebra of the superstring on S^5 with the AdS-time is shown on the light-like plane. To this end we use the geometrical method of which consistency is guaranteed by the classical Yang-Baxter equation. The Dirac method does not work, there being constraints which contain first-class and second-class and one can disentangle with each other keeping the isometry hardly.

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The AdS/CFT correspondence between the type IIB string theory on $AdS_5 \times S^5$ and the $N = 4$ supersymmetric QCD is one of the subjects which have been discussed with great interest in the last decade. The study of such string/QCD duality has a long history going back to the late 70's[1]. The AdS/CFT correspondence[2] is the most clear-cut assertion of string/QCD duality given ever since. Correlation functions were extensively studied on both sides of the correspondence to check this. Later on people discovered a remarkable relationship between the $N = 4$ supersymmetric QCD and the Heisenberg spin-chain systems. Namely they claimed that chains of scalar fields of the former theory can be identified with spin-chains of the latter by calculating the anomalous dimension and the energy for the respective chains[3]. It is no doubt that integrablity of both chain systems is involved behind this phenomenon. From the AdS/CFT duality we are then lead to demand a corresponding integrability on the string side. It is well-known that the XXX Heisenberg spin-chain equivalent to the $O(3)$ non-linear σ-model at the thermodynamical limit[4]. Therefore it is quite natural to consider the the type IIB string theory on $AdS_5 \times S^5$ as a non-linear σ-model and suspect its integrable structure. Indeed many works have been done about integrability of non-linear σ-models on S^5 and AdS_5[5, 6].

Exchange algebrae based on the Yang-Baxter equation are also characteristic for integrable systems. This aspect of the type IIB string theory on $AdS_5 \times S^5$[7] is currently attracting much attention in connection with higher-loop integrability of $N = 4$ supersymmetric QCD[8]. In this letter we will shed a light on the subject from a different angle. The following action is a part of the type IIB string theory on $AdS_5 \times S^5$

$$S = \int d^2\xi \Big[\sum_{a=1}^{6} \partial^\mu M^a \partial_\mu M^a - \partial^\mu X^0 \partial_\mu X^0 \Big], \tag{1}$$

where X^0 is the AdS-time and M^a are fields constrained on S^5 by $\sum_{a=1}^6 M^a M^a = 1$. It is supplimented by the Virasoro constraint

$$\partial_\pm M^a \partial_\pm M^a = \partial_\pm X^0 \partial_\pm X^0.$$

With a gauge $X^0 \propto t$ the action (1) becomes the non-linear σ-model on S^5[9]. The aim of this letter is to show, though at the classical level, in a rather simple way the exchange algebra for the non-linear σ-model on S^5

$$\begin{aligned} &\{M^a(\xi^+, \xi^-), M^b(\xi^+, \xi'^-)\} \\ &\quad = -\frac{1}{4}\Big[\theta(\xi^- - \xi'^-) r^+ + \theta(\xi'^- - \xi^-) r^-\Big]^{ab}_{cd} M^c(\xi^+, \xi^-) M^d(\xi^+, \xi'^-). \end{aligned} \tag{2}$$

Here $\{\ ,\ \}$ is the Poisson bracket on the light-cone plane $\xi^+ = const.$ and $r^\pm$ are the classical r-matrices given in the fundamental representation of $SO(6)$, which satisfy the classical Yang-Baxter equation. As a deformation of (2) we might think of a quantum exchange algebra

$$\begin{aligned} &M^a(\xi^+, \xi^-) M^b(\xi^+, \xi'^-) \\ &\quad = \Big[\theta(\xi^- - \xi'^-) R^+ + \theta(\xi'^- - \xi^-) R^-\Big]^{ab}_{cd} M^d(\xi^+, \xi'^-) M^c(\xi^+, \xi^-), \end{aligned}$$

by extending the classical r-matrices to quantum R-matrices as

$$R^{\pm} = 1 \otimes 1 + \hbar r^{\pm} + O(\hbar^2).$$

with $\hbar = -\frac{1}{4}$. The similar classical exchange algebrae in the canonical formalism were discussed for the 2-dimensional effective gravity in the literature[10]. We will proceed the arguments exactly in the same way as for that case.

We shall consider non-linear σ-models on the coset space G/H, which are given by the action

$$S = \int d^2\xi \ \mathcal{L} = \frac{1}{2} \int d^2\xi \ \eta^{\mu\nu} g_{ij}(X) \partial_\mu X^i \partial_\nu X^j, \tag{3}$$

in the 2-dimensional flat world-sheet. The energy-momentum tensor and the isometry current are respectively given by

$$T_{\mu\nu} = g_{ij}(X) \partial_\mu X^i \partial_\nu X^j - \eta_{\mu\nu} \mathcal{L}, \qquad J^A_{\ \mu} = R^A_{\ i}(X) \partial_\mu X^i.$$

Here R^{Ai} are the Killing vectors of the coset space G/H, which non-linearly realize the Lie-algebra of G as

$$R^{Ai}_{\ \ ,j} R^{Bj} - R^{Bi}_{\ \ ,j} R^{Aj} = f^{AB}_{\ \ \ C} R^{Ci}, \tag{4}$$

and satisfy the Killing equations

$$R^A_{\ \{i;j\}} \equiv R^A_{\ i;j} + R^A_{\ j;i} = 0. \tag{5}$$

The non-linear σ-models (3) have conformal invariance and isometry so that

$$T_{+-} = 0, \qquad \partial_+ T_{--} = 0, \qquad \partial_- T_{++} = 0, \qquad \partial_+ J^A_- + \partial_- J^A_+ = 0,$$

due to the equation of motion

$$\nabla^\mu \partial_\mu X^i \equiv \partial^\mu \partial_\mu X^i + \Gamma^i_{jk} \partial^\mu X^j \partial_\mu X^k = 0. \tag{6}$$

We study the canonical structure of the models on the light-like plane setting up the Poisson brackets. It can be done by the geometrical method formulated in [10]. Namely we may set up the Poisson brackets $\{X^i, X^j\}$ on the light-like plane $\xi^+ = const.$ so as to be able to correctly reproduce the diffeomorphism and the isometry transformations as

$$\begin{aligned} \delta_{diff} X^i(\xi^+, \xi^-) &\equiv \epsilon(\xi^-) \partial_- X^i(\xi^+, \xi^-) \\ &= \int d\zeta^- \epsilon(\zeta^-) \{X^i(\xi^+, \xi^-), T_{--}(X(\xi^+, \zeta^-))\}, \end{aligned} \tag{7}$$

$$\begin{aligned} \delta_{iso} X^i(\xi^+, \xi^-) &\equiv \epsilon_A R^{Ai}(X(\xi^+, \xi^-)) \\ &= \int d\zeta^- \epsilon_A \{X^i(\xi^+, \xi^-), J^A(X(\xi^+, \zeta^-))\}. \end{aligned} \tag{8}$$

Here $\epsilon(\xi^-)$ and ϵ_A are respectively local and global parameters of the transformations. It turns out that they are given in the form

$$\begin{aligned}\{X^i(\xi^-), X^j(\eta^-)\} &= -\{X^j(\eta^-)\, X^i(\xi^-)\} \\ &= -\frac{1}{4}\theta(\xi^- - \eta^-) t^+{}_{AB} R^{Ai}(X(\xi^-)) R^{Bj}(X(\eta^-)) \\ &\qquad + \frac{1}{4}\theta(\eta^- - \xi^-) t^+{}_{AB} R^{Aj}(X(\eta^-)) R^{Bi}(X(\xi^-)).\end{aligned} \tag{9}$$

The notation of this formula is as follows. $\theta(\xi^-)$ is the step function and $R^{Ai}(X(\xi^+))$ are the Killing vectors of the coset space G/H, given by (4) and (5). The world-sheet coordinate ξ^+ in $X^i(\xi^+, \xi^-)$ was omitted to avoid a unnecessary complication. $t^+{}_{AB}$ is the most crucial part for our argument in this letter. It is a set of the coefficients taken from the classical r-matrices

$$r^\pm = \sum_{\alpha \in R} sgn\ \alpha E_\alpha \otimes E_{-\alpha} + \sum_{A,B} t_{AB} T^A \otimes T^B \equiv t^\pm{}_{AB} T^A \otimes T^B, \tag{10}$$

with T^A the generators of the group G given in the Chevallay basis as $\{E_{\pm\alpha}, H_\mu\}$, $t^+{}_{AB}$ the corresponding Killing metric and $sgn\ \alpha = \pm$ according as the roots are positive or negative. The r-matrix satisfies the classical Yang-Baxter equation[11]

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{11}$$

which guarantees the Jacobi identities for the Poisson brackets(9), as will be shown later.

First of all we will show that the diffeomorphism (7) can be reproduced by using the Poisson brackets (9). A little algebra yields

$$\begin{aligned}\{X^i(\xi^-), T_{--}(X(\zeta^-))\} &= \delta(\xi^- - \zeta^-) \partial_- X^j(\zeta^-) g_{jk}(\zeta^-) t_{AB} R^{Ai}(\xi^-) R^{Bk}(\zeta^-) \\ &\quad - \frac{1}{4}\theta(\xi^- - \zeta^-) t^+{}_{AB} R^{Ai}(\xi^-) R^B{}_{\{j;k\}}(\zeta^-) \partial_- X^j(\zeta^-) \partial_- X^k(\zeta^-) \\ &\quad + \frac{1}{4}\theta(\zeta^- - \xi^-) t^+{}_{AB} R^A{}_{\{j;k\}}(\zeta^-) R^{Bi}(\xi^-) \partial_- X^j(\zeta^-) \partial_- X^k(\zeta^-).\end{aligned}$$

This becomes

$$\{X^i(\xi^-), T_{--}(X(\zeta^-))\} = \delta(\xi^- - \zeta^-) \partial_- X^i(\zeta^-), \tag{12}$$

by the Killing equations (5) and the property of the Killing vectors[12]

$$t_{AB} R^{Ai} R^{Bj} = g^{ij}.$$

With (12) the diffeomorphism (7) is correctly reproduced. Similarly one can check that the isometry transformations (8) can be reproduced.

We remark that the above demonstration would work even if the Poisson brackets (9) were simply given with the Killing metric t_{AB} in place of $t^+{}_{AB}$. The specfic choice of $t^+{}_{AB}$ given by (10) is required to show the Jacobi identities

$$\begin{aligned}Q^{ijk} \equiv &\{X^i(\xi^-), \{X^j(\zeta^-), X^k(\eta^-)\}\} + \{X^j(\zeta^-), \{X^k(\eta^-), X^i(\xi^-)\}\} \\ &+ \{X^k(\eta^-), \{X^i(\xi^-), X^j(\zeta^-)\}\} = 0.\end{aligned} \tag{13}$$

To show it, we calculate the quantities Q^{ijk} by (9) assuming that $\xi^- > \zeta^- > \eta^-$. We then use the the Lie-Algebra for the Killing vectors (4) to find

$$
\begin{aligned}
Q^{ijk} = & -\frac{1}{4} t^+{}_{AB} t^+{}_{CD} \Big[f^{AC}{}_E R^{Ei}(\xi^-) \Big] R^{Dj}(\zeta^-) R^{Bk}(\eta^-) \\
& - \frac{1}{4} t^+{}_{AB} t\ {}_{CD} R^{Ci}(\xi^-) \Big[f^{AD}{}_E R^{Ej}(\zeta^-) \Big] R^{Bk}(\eta^-) \\
& - \frac{1}{4} t^+{}_{AB} t^+{}_{CD} R^{Ci}(\xi^-) R^{Aj}(\zeta^-) \Big[f^{BD}{}_E R^{Ek}(\eta^-) \Big].
\end{aligned}
\tag{14}
$$

Note that eq. (4) can be put in the form

$$
R^{Ci} \propto f_{AB}{}^C R^{Ai}{}_{,j} R^{Bj},
$$

by using $f_{AB}{}^C f^{AB}{}_D \propto \delta^C_D$. Replace all the Killing vectors in (14) by this formula. Then it becomes

$$
\begin{aligned}
Q^{ijk} \propto \frac{1}{4} t^+{}_{AB} t^+{}_{CD} \Big(& ([T^A, T^C])_{EF} \otimes (T^B)_{GH} \otimes (T^D)_{KL} \\
& + (T^A)_{EF} \otimes ([T^B, T^C])_{GH} \otimes (T^D)_{KL} \\
& + (T^A)_{EF} \otimes (T^C)_{GH} \otimes ([T^B, T^D])_{KL} \Big) \\
& \times R^{Ei}{}_{,l}(\xi^-) R^{Fl} \cdot (\xi^-) R^{Gj}{}_{,m}(\zeta^-) R^{Hm}(\zeta^-) \cdot R^{Ki}{}_{,n}(\eta^-) R^{Ln}(\eta^-).
\end{aligned}
$$

This is vanishing due to the classical Yang-Baxter equation (11) so that the Jacobi identities (13) are satisfied.

Thus it has been shown that the Poisson brackets (9) are correct for the canonical formalism of the non-linear σ-models (3) on the light-like plane. We will use this to find the classical exchange algebra (2) in the non-linear σ-models. Our claim is that such an algebra exists if the models admit local composite fields $M^a(X), a = 1, 2 \cdots, d$ which change as

$$
\begin{aligned}
\delta M^a(X) & \equiv \epsilon_A R^{Ai}(X) M^a{}_{,i}(X) \\
& = \epsilon_A (T^A)^a{}_b M^b,
\end{aligned}
\tag{15}
$$

under the isometry transformation $\delta X^i = \epsilon_A R^{Ai}(X)$. That is, the composite fields $M^a(X)$ belong to a d-dimensional representation of the isometry group G. It depends on the type of the coset space G/H whether such composite fields exist or not. We proceed with the argument assuming the existence of them. By using the Poisson brackets (9) we find that

$$
\begin{aligned}
\{M^a(X(\xi^-)), M^b(X(\zeta^-))\} & = M^a{}_{,i}(X(\xi^-)) \{X^i(\xi^-), M^b(X(\zeta^-))\} \\
& = M^a{}_{,i}(X(\xi^-)) \{X^i(\xi^-), X^j(\zeta^-)\} M^b{}_{,j}(X(\zeta^-)) \\
& = -\frac{1}{4} \theta(\xi^- - \zeta^-) t^+{}_{AB} (T^A)^a{}_c (T^B)^b{}_d M^c(X(\xi^-))) M^d(X(\zeta^-)) \\
& + \frac{1}{4} \theta(\zeta^- - \xi^-) t^+{}_{AB} (T^A)^a{}_c (T^B)^b{}_d M^c(X(\zeta^-))) M^d(X(\xi^-)).
\end{aligned}
$$

By using the property ${t^+}_{AB} = -{t^+}_{BA}$ and the r-matrices (10) we get the classical exchange algebra (2).

We apply the above arguments to the case of the non-linear σ-model on the coset space $SO(6)/SO(5)(= S^5)$. For this model there exsits a set of composite fields M^a transforming as (15) in the fundamental representation of $SO(6)$. They are constrained by $\sum_{a=1}^{6} M^a M^a = 1$ and are parametraized as

$$\begin{aligned}
M^1(X) &= \cos X^1, \\
M^2(X) &= \sin X^1 \cos X^2, \\
M^3(X) &= \sin X^1 \sin X^2 \cos X^3, \\
M^4(X) &= \sin X^1 \sin X^2 \sin X^3 \cos X^4, \\
M^5(X) &= \sin X^1 \sin X^2 \sin X^3 \sin X^4 \cos X^5, \\
M^6(X) &= \sin X^1 \sin X^2 \sin X^3 \sin X^4 \sin X^5.
\end{aligned}$$

By this parametrization we rewrite the $SO(6)$ generators to obtain the Killing vectors $R^{[ab]i}$ as

$$iM^a \frac{\partial}{\partial M_b} - iM^b \frac{\partial}{\partial M_a} = R^{[ab]i} \frac{\partial}{\partial X_i}.$$

Therefore the whole arguments leading us to the exchange algebra (2) go through for this special case. As the result we obtain the exchange algebra (2) with the r-matrices given in the fundamental representation of $SO(6)$.

The reader might think of studying the Poisson brackets (9) by using the Dirac method. However we can see that it does not work. For the model (3) we have

$$\pi_i \equiv \frac{\delta \mathcal{L}}{\delta \partial_+ X^i} = g_{ij} \partial_- X^j,$$

which lead us to a set of constraints

$$\phi_i(\xi) \equiv \pi_i - g_{ij} \partial_- X^j = 0. \tag{16}$$

They are typical for the Lagrangian which is homogeneous of the first degree in the velocities $\partial_+ X^i$. According to the Dirac method[13] the Hamiltonian is merely given by

$$\mathcal{H} = \int d\xi^- \lambda^i(\xi) \phi_i(\xi),$$

with Lagrangian multipliers $\lambda^i(\xi)$. Setting the Poisson brackets as $\{X^i(\xi^-), \pi_j(\eta^-)\}_P = \delta^i_j \delta(\xi^- - \eta^-)$ on the light-like plane $\xi^+ = const.$ we find that

$$\begin{aligned}
C_{ij}(\xi^-, \eta^-) &\equiv \{\phi_i(\xi^-), \phi_j(\eta^-)\}_P \\
&= -g_{ij}(\xi^-) \partial^\xi_- \delta(\xi^- - \eta^-) - g_{ik,j}(\xi^-) \partial_- X^k(\xi^-) \delta(\xi^- - \eta^-) \\
&\quad + g_{ji}(\eta^-) \partial^\eta_- \delta(\xi^- - \eta^-) + g_{jk,i}(\eta^-) \partial_- X^k(\eta^-) \delta(\xi^- - \eta^-).
\end{aligned} \tag{17}$$

Here again the coordinate ξ^+ was omitted from $X^i(\xi^+, \xi^-), \pi_i(\xi^+, \xi^-), \phi_i(\xi^+, \xi^-)$ and $g_{ij}(\xi^+, \xi^-)$. They look like second-class constraints. But they are a mixture of first- and second-class constraints as follows. The consistency of the constraints requires that

$$\{\phi_i(\xi^-), \mathcal{H}\} = \int d\eta^- \lambda^j(\eta)\{\phi_i(\xi^-), \phi_j(\eta^-)\} = 0, \tag{18}$$

which becomes $\nabla_- \lambda^i(\xi^+, \xi^-) = 0$. With $\lambda^i = \partial_+ X^i$ they are satisfied by means of the equation of motion (6). Therefore the quantity $C_{ij}(\xi^-, \eta^-)$ is not invertible. It means that the constraints (16) contain first-class ones.

A similar problem can be seen in the Green-Schwarz formulation of superparticle or superstring. The Green-Schwarz superparticle[14] is given by

$$S \equiv \int dt \mathcal{L} = \frac{1}{2} \int dt e^{-1} (\dot{X}^M - i\bar{\Theta}\gamma^M \dot{\Theta})(\dot{X}_M - i\bar{\Theta}\gamma_M \dot{\Theta}).$$

The problematic constraint takes the form

$$\phi \equiv \pi_\Theta - i\bar{\Theta}\not{p} = 0, \tag{19}$$

with $\pi_\Theta = \frac{\delta \mathcal{L}}{\delta \dot{\Theta}}$ and $p_M = \frac{\delta \mathcal{L}}{\delta \dot{X}^M}$. A simple algebra gives

$$C_{\alpha\beta} \equiv \{\phi_\alpha, \phi_\beta\}_{+P} = -2i(\gamma^0 \not{p})_{\alpha\beta}.$$

We have $p^2 = 0$ as the equation of motion for e. Therefore $C_{\alpha\beta}$ is not invertible. Thus constraints (19) are a mixture of first-class and second-class. We do not know how to disentangle the two classes of constraints with each other in a Lorentz-covariant manner. This is why the Green-Schwarz superparticle can not be quantized in a covariant way.

We do not know either how to do the disentanglement of the constraints (16) in a isometrical way. Therefore the Dirac method hardly works for the non-linear σ-models on the light-like plane. The geometrical arguments in this letter gives an alternative way to the Dirac method. But note that

$$\begin{aligned}
&\int d\eta^- C_{ij}(\xi^-, \eta^-)\{X^j(\eta^-), X^k(\zeta^-)\} \\
&\qquad = \delta_i^k \delta(\xi^- - \zeta^-) + \frac{1}{2}\theta(\xi^- - \zeta^-) t^+_{AB} R^A_{\ i;j}(\xi^-) \partial_- X^j(\xi^-) R^{Bk}(\zeta^-),
\end{aligned}$$

with the Poisson brackets (9) and $C_{ij}(\xi^-, \eta^-)$ given by (17). For the special case when the target space is flat, the second term in the *r.h.s.* vanishes owing to the equation of motion, which is solved by holomorphic functions $X^i(\xi^+, \xi^-) = f^i(\xi^+) + g^i(\xi^-)$. Then $C_{ij}(\xi^-, \eta^-)$ becomes invertible as

$$\Big[C(\xi^-, \eta^-)^{-1}\Big]^{ij} \equiv \{X^i(\xi^-), X^j(\eta^-)\} = -\frac{1}{4}\delta^{ij}\Big[\theta(\xi^- - \eta^-) - \theta(\eta^- - \xi^-)\Big].$$

Putting this in other words, eq. (18) has no other solution than the trivial ones $\lambda^i = \partial_+ X^i = 0$ for this case, so that the constraints (16) do not contain first-class ones at all. Consequently the Dirac method works in the standard way.

In this letter we have shown the exchange algebra of non-linear σ-models on the light-like plane $\xi^+ = const.$, setting the canonical structure by the geometrical arguments. Of course the exchange algebra of non-linear σ-models can be discussed also on the space-like plane with $t = const.$. The usual canonical method does work well giving an exchange algebra for the transfer matrix, but not for local fields like M^a[6].

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